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REPRESENTATION OF THE SOLUTIONS OF THE NAVIER-STOKES SYSTEM NEAR THE CONTACT CHARACTERISTIC*

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It is shown, for the system of Navier-Stokes equations describing the flows of a viscous, heat conducting compressible fluid, that the contact surface is a characteristic of unit multiplicity. The conditions are obtained, which must be specified, for the unique solvability of the corresponding Cauchy problem. It is shown that if the initial data of the problem are analytic, then so is its solutions, and an algorithm is given for constructing it. A transport equation is written out for a weak shock at the contact surface. A solution of the transport equation is given for one-dimensional, plane-symmetric flows, and the form of the first coefficients of the series describing the flow. The time exponent is revealed, which determines the process of smoothing the small perturbations near the corresponding contact surfaces. Solutions decaying with time are constructed in the form of series in powers of this exponent. The first terms of the series are periodic functions of the spatial variable. Two fundamental frequencies can be singled out in the periodic terms, and the frequencies are inversely proportional to the viscosity. The possibility of corresponding oscillations appearing in a flow of viscous gas is discussed.

1. We consider the system of Navier-Stokes equations

$$\begin{gathered}
\frac{\partial \rho}{\partial t}+\mathbf{V}\cdot\nabla\rho+\rho\operatorname{div}\mathbf{V}=0 \\
\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|^*\right)+c_1{}^2\nabla\rho+b_1\nabla T=\rho\mathbf{g}+ \\
(\operatorname{div}\mathbf{V})\left(\nabla\mu'-\frac{2}{3}\nabla\mu\right)+\nabla\mu\left(\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|^*+\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|\right)+ \\
\left(\mu'+\frac{1}{3}\mu\right)\nabla(\operatorname{div}\mathbf{V})+\mu\Delta\mathbf{V} \\
c_v\rho\left(\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T\right)+s_1\operatorname{div}\mathbf{V}=\varkappa\Delta T+\nabla\varkappa\cdot\nabla T+\Phi+Q
\end{gathered} \tag{1.1}$$

for the flows of a viscous, heat conducting compressible fluid /1, 2/. Here t is the time, x_1, x_2, x_3 are the spatial coordinates, ρ is the density, v_1, v_2, v_3 are the Cartesian projections of the vector $\mathbf{V}$ of velocity of the medium, T is temperature, $\mathbf{g}$ is the external mass forces vector, μ, μ' are the dynamic and volume coefficients of viscosity, $\varkappa$ is the thermal donductivity, Q is the intensity of heat sources or sinks, Φ is the dissipation function, $\|\partial v_\alpha/\partial x_\beta\|$ is a Jacobi matrix, $\|\partial v_\alpha/\partial x_\beta\|^*$ is its transpose, $\Delta, \operatorname{div}, \nabla$ are the Laplace, divergence and gradient operators, a dot denotes a scalar product, the vectors are regarded as line vectors, and the product of a vector and a matrix is obtained according to the rules of matrix multiplication. When deriving system (1.1), we assumed that ρ, T were chosen as the independent thermodynamic parameters and the equations of state satisfied the fundamental thermodynamic identity. For this reason we assume that

$$\mu,\ \mu',\ \varkappa,\ p,\ e \tag{1.2}$$

are given as functions of ρ, T /2/ (p is the pressure and e denotes the internal energy). Then we have

$$c_1=\frac{\partial p}{\partial \rho},\quad b_1=\frac{\partial p}{\partial T},\quad c_v=\frac{\partial e}{\partial T},\quad s_1=p-\rho^2\frac{\partial e}{\partial \rho}=Tb_1$$

when $\mu'=0$; $\mu, \varkappa=\mathrm{const}>0$ we have for system (1.1), in the case of an ideal (real) polytropic gas with the equations of state

$$p=R\rho T,\ e=c_vT;\ R, c_v=\mathrm{const}>0 \tag{1.3}$$

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the theorems of existence of solutions of the Cauchy and the boundary value problems, local in the multidimensional case, and global in the one-dimensional plane-symmetric case. Exact formulations and detailed references are given in /3/.

In the general case, when the parameters (1.2) depend on ρ, T, it was shown in /4/ that for small t a solution of the multidimensional Cauchy problem exists if the initial distribution of temperature and the values of $\mu, \varkappa$ are strictly separated from zero. In the case when $\mu = \mu' = \varkappa = 0$, system (1.1) has five characteristics: two sonic characteristics and a contact characteristic of multiplicity three /5/. When $\mu \neq 0, \varkappa \neq 0$ (1.1) is of mixed type and, as is shown below, when $\mu > 0, \mu' \geqslant 0, \varkappa > 0$, the contact surface in compressible flows is a characteristic of multiplicity one.

The presence of a characteristic makes it possible to join different solutions across a weak discontinuity and employ to this end, when the initial data of the problem are analytic, the method of characteristic series /6, 7/. The problem of why system (1.1) has a characteristic and the possibility of constructing a local analytic solution in its neighbourhood, is dealt with as in /8/.

Let a surface in $(t, \mathbf{x})$ space be defined by the equation

$$x_1 = a(t, x_2, x_3) \tag{1.4}$$

where the function a is assumed to be analytic in some neighbourhood of the point $(t = t_0,\ x_2 = x_{20},\ x_3 = x_{30})$ and $a(t_0, x_{20}, x_{30}) = x_{10}$. On making the change of variables

$$z = x_1 - a(t, x_2, x_3),\ \xi = x_2,\ \zeta = x_3,\ t' = t \tag{1.5}$$

the surface (1.4) becomes the coordinate plane $z = 0$. The derivatives in t, ξ, ζ (a prime accompanying t is omitted) will be internal for this plane, and those in z will be outward, and $\partial/\partial z = \partial/\partial x_1$. The form which system (1.1) takes after the change of variables (1.5) is cumbersome, and will therefore be omitted. Henceforth, we shall also call this system, for brevity, system (1.1). Since the leading outward derivatives of the unknown functions $\mathbf{U} = \{\rho, \mathbf{V}, T\}$ will, in this system, be $\rho_z, \mathbf{V}_{zz}, T_{zz}$ respectively, it follows that the initial conditions in the formulation of the Cauchy problem on the surface (1.4) must be

$$z = 0,\ \rho = \rho_0,\ \mathbf{V} = \mathbf{V}_0,\ \mathbf{V}_z = \mathbf{V}_1,\ T = T_0,\ T_z = T_1 \tag{1.6}$$

(the right-hand sides of Eqs.(1.6) are given functions of t, ξ, ζ).

The only leading outward derivative appearing in the equation of continuity is ρ_z, which is linear and has the coefficient $b = v_1 - a_\xi v_2 - a_\zeta v_3 - a_t$. When $a, \mathbf{V}_0$ are given, the condition $b = 0$, i.e.

$$v_{10} - a_\xi v_{20} - a_\zeta v_{30} - a_t = 0 \tag{1.7}$$

is equivalent to the statement that (1.4) is a contact surface consisting of the trajectories of the particles of the medium which form, at the instant $t = t_0$, the surface $x_1 = a(t_0, x_2, x_3)$.

If $b = 0$ and the functions (1.4) and (1.6) are given, then the equation of continuity at $z = 0$ will take the form

$$\rho_{0t} + v_{20}\rho_{0\xi} + v_{30}\rho_{0\zeta} + \rho_0 (v_{11} - a_\xi v_{21} - a_\zeta v_{31} + v_{20\xi} + v_{30\zeta}) = 0 \tag{1.8}$$

Since (1.8) does not contain any leading outward derivatives, it follows that it represents an additional relation which is imposed on the initial conditions (1.6). Therefore the Cauchy problem (1.1), (1.6) with condition (1.7) will represent the characteristic Cauchy problem and relation (1.8) will be a necessary condition for its solvability /8/. In what follows, we shall assume that when the functions (1.4), (1.6) are given, conditions (1.7), (1.8) hold. In particular, this will be the case if (1.4), (1.6) have been obtained in the corresponding manner from any solution of system (1.1).

The leading outward derivative of $T - T_{zz}$ appears only in the energy equation, and the coefficient preceding it is equal to $\varkappa A$, $A = 1 + a_\xi^2 + a_\zeta^2$. Therefore when

$$\varkappa(\rho_0, T_0) > 0 \tag{1.9}$$

the energy equation can be solved for T_{zz}. $\mathbf{V}_{zz}$ will not appear on the right-hand side of the equation obtained, and ρ_z will be there only if $\varkappa$ depends on ρ. The physical meaning of the function $\varkappa$ is taken into account in condition (1.9), although formally it would be sufficient to insert the inequality sign, there $v_{\alpha zz}$ $(\alpha = 1, 2, 3)$ are described by a system of linear algebraic equations with the determinant $d = (\mu' + 4\mu/3)\mu^2 A^3$, ρ_z appears only in the free terms of this system and is also linear. Therefore, when the following conditions hold (the physical meaning of the functions μ, μ' is taken into account):

$$\mu(\rho_0, T_0) > 0,\ \mu'(\rho_0, T_0) \geqslant 0 \tag{1.10}$$

$v_{\alpha zz}$ can be expressed in terms of $\mathbf{U}, \rho_z, \mathbf{V}_z, T_z$ and their inner derivatives. The resulting expressions for $v_{\alpha zz}$ are then substituted into the equation of continuity differentiated once with respect to z. the expression obtained in this manner contains all second-order derivatives in linear form, the only leading outward derivative present is ρ_{zz} with coefficient b, and the coefficient in front of ρ_{zt} is equal to unity.

This reduces the characteristic Cauchy problem formulated above, to its standard form /8/: 1) in order to obtain a unique solution of problem (1.1), (1.6) where the relations (1.7)-(1.10) hold, it is sufficient to specify another additional condition

$$\rho(z, t_0, \xi, \zeta) = \rho_{01}(z, \xi, \zeta), \rho_{01}(0, \xi, \zeta) = \rho_0(t_0, \xi, \zeta) \tag{1.11}$$

with an arbitrary function ρ_{01} correlated with ρ_0; 2) when the functions g, Q (1.2), (1.6), (1.11) are analytic in the neighbourhood of the point $(t_0, \mathbf{x}_0, \mathbf{U}_0)$, the solution of problem (1.1), (1.6), (1.11) will be analytic functions.

The solution can be written in the form of locally convergent series in powers of z. The coefficients of the series depend on t, ξ, ζ and can be found with the help of recurrence methods as follows. The equation of continuity is differentiated $k+1$ times $(k \geqslant 0)$ with respect to z, the equations of motion and energy k times, z is equated to zero, and the initial conditions and the coefficients already obtained are substituted into the equations. The differentiated equations of motion, regarded as an algebraic system with non-zero coefficients, yield the components of the vector $\mathbf{V}_{k+2} = \partial^{k+2}\mathbf{V}(0, t, \xi, \zeta)/\partial z^{k+2}$, which depend only on $\rho_{k+1} = \partial^{k+1}\rho(0, t, \xi, \zeta)/\partial z^{k+1}$ and the preceding coefficients of the series. The expressions obtained are substituted into the differentiated equation of continuity, which thereby becomes a first-order partial differential equation for ρ_{k+1} whose coefficients and right-hand side depend only on the preceding coefficients of the series; the coefficient in front of $\partial\rho_{k+1}/\partial t$ is equal to unity; when $t = t_0$, the initial condition for ρ_{k+1} is found from relation (1.11) differentiated with respect to z $k+1$ times. Thus we have obtained ρ_{k+1} as a solution of the corresponding Cauchy problem for the first-order partial differential equations. We then find $\mathbf{V}_{k+2}$ from the differentiated equations of motion. Finally, the differentiated energy equation regarded as a linear algebraic equation with a non-zero coefficient preceding the unknown, yields $T_{k+2} = \partial^{k+2}T(0, t, \xi, \zeta)/\partial z^{k+2}$.

All differential equations for ρ_{k+1} $(k \geqslant 0)$ are, by virtue of the characteristic features of (1.1), linear. The first equation is a so-called transport equation, since it describes the behaviour of ρ_1

$$\begin{gathered}\rho_{1t} + v_{20}\rho_{1\xi} + v_{30}\rho_{1\zeta} + [2(v_{11} - a_\xi v_{21} - a_\zeta v_{31}) + v_{20\xi} + v_{30\zeta}]\rho_1 + \\ \rho_0(v_{12} - a_\xi v_{22} - a_\zeta v_{32} + v_{21\xi} + v_{31\zeta}) + v_{21}\rho_{0\xi} + v_{31}\rho_{0\zeta} = 0\end{gathered} \tag{1.12}$$

The quantities $v_{\alpha 2} = v_{\alpha zz}(0, t, \xi, \zeta)$ in the above equation are replaced by the corresponding expressions obtained from the equations of motion when $z = 0$. We note that the coefficients of Eq.(1.12) depend on μ, μ' and are independent of $\varkappa$. For certain relations between the functions (1.6) (these relations, as well as those for $v_{\alpha 2}$, are cumbersome and are therefore omitted) the equation for ρ_1 becomes homogeneous and the quantity ρ_1 in this case is either always equal to, or always different from zero.

The equations of motion yield a linear relation between v_{1zz} and ρ_z, with coefficients different from zero in the case when conditions (1.10) and $c_1^2(\rho_0, T_0) > 0$ all hold. Therefore we can specify, in place of (1.11), as the additional condition ensuring the uniqueness of the solution, $v_1(z, t_0, \xi, \zeta) = v_{01}(z, \xi, \zeta)$ with an arbitrary function v_{01}, but satisfying the conditions of matching with the initial data (1.6): $v_{01}(0, \xi, \zeta) = v_{10}(t_0, \xi, \zeta)$, $v_{01z}(0, \xi, \zeta) = v_{11}(t_0, \xi, \zeta)$. When different solutions are "joined" together at the contact surface, weak discontinuities are, in general, present, beginning with the derivatives $\rho_z, \mathbf{V}_{zz}, T_{zz}$. Therefore the mass, momentum and energy fluxes in the "joined" solution depending on $\mathbf{U}, \mathbf{V}_z, T_z$ will be continuous at the contact surface. All these arguments imply that when conditions (1.9), (1.10) hold, system (1.1) has, apart from the contact characteristic, no other characteristics of the form (1.4).

Using the methods of /9, 10/ we can show that the end points $t_{1*} < t < t_{2*}$ of the domain of convergence of the series from the analytic function $\mathbf{U}$ of the variable t (when $t \to t_{i*}$ and ξ, ζ is fixed, the radius of convergence of the series tends to zero as some positive powder of $|t - t_{i*}|$) are the points nearest to $t = 0$ at which the function (1.6) ceases to be analytic ρ_1, $1/(\varkappa A)$, $1/d$. In particular, if the functions listed above are analytic for all t, then the domain of convergence of the series for $\mathbf{U}$ will be unbounded in t and the radius of convergence will tend to zero as $|t|$ increases.

2. We will discuss certain properties of the solutions of the characteristic Cauchy problem with the data specified at the contact surface, and the possibility of using these solutions in specific problems. We shall deal with the case of one-dimensional plane symmetric gas flows with equations of state (1.3) and the coefficient of viscosity and thermal conductivity $\mu = \mu_0 T^\omega$, $\mu' = 0$, $\varkappa = \varkappa_0 T^\lambda$; $\mu_0, \varkappa_0 = \text{const} > 0$, $\omega, \lambda = \text{const} \geqslant 0$, without external forces and heat sources.

Using certain positive constants L, ρ_*, u_*, T_*, specified by the formulation of the specific problem, we can write system (1.1), using standard methods, in terms of dimensionless variables, thus

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{2.1}$$

$$\rho(u_t + uu_x) + \frac{1}{M^2\gamma}(T\rho_x + \rho T_x) = \frac{4}{3\,\mathrm{Re}} T^{\omega-1}(\omega T_x u_x + T u_{xx})$$

$$\rho(T_t + uT_x) + (\gamma-1)\rho T u_x =$$

$$(\gamma-1)\frac{4M^2\gamma}{3\,\mathrm{Re}} T^{\omega} u_x^2 + \frac{\gamma}{\mathrm{Pr}\,\mathrm{Re}} T^{\lambda-1}(\lambda T_x^2 + T T_{xx})$$

$$x = x_1, \quad u = v_1, \quad M^2 = \frac{u_*^2}{c_v(\gamma-1)T_*\gamma} = \frac{u_*^2}{c_*^2},$$

$$\mathrm{Re} = \frac{L\rho_* u_*}{\mu_*}, \quad \mathrm{Pr} = \frac{c_v\gamma\mu_*}{\varkappa_*}.$$

$$\gamma = 1 + R/c_v, \quad \mu_* = \mu_0 T_*^{\omega}, \quad \varkappa_* = \varkappa_0 T_*^{\lambda}$$

Let the contact surface, which can be regarded as a trajectory of motion of an impermeable piston, be described by the equation

$$x = a(t) \tag{2.2}$$

We introduce, in agreement with the results obtained above, the independent variable $z = x - a(t)$ and specify for $z = 0$, i.e. on the contact surface, the values of the gas-dynamic parameters and the first outward derivatives $(\partial/\partial z = \partial/\partial x)$ for the velocity and temperature

$$z = 0, \ \rho = \rho_0(t), \ u = u_0(t), \ u_z = u_1(t), \ T = T_0(t), \quad T_z = T_1(t) \tag{2.3}$$

The condition that (2.2) is a contact surface (relation (1.7)) takes the form $u_0(t) = a'(t)$. The necessary condition for problem (2.1), (2.3) to have a solution (relation (1.8)) becomes $u_1(t) = -\rho_0'(t)/\rho_0(t)$. Here and henceforth we assume that $\rho(x,t) > 0$. The unique solution of problem (2.1), (2.3) is found using the additionally specified distribution of the density at the initial instant near the point $x = x_0 = a(0)$

$$\rho(x,t)|_{t=t_0} = \rho_{01}(x), \ \rho_{01}(x_0) = \rho_0(0) \tag{2.4}$$

The solution of problem (2.1), (2.3), (2.4) is written in the form of a locally converging series

$$\mathrm{U}(z,t) = \sum_{k=0}^{\infty} \mathrm{U}_k(t)\frac{z^k}{k!}, \quad \mathrm{U}_k(t) = \left.\frac{\partial^k \mathrm{U}}{\partial z^k}\right|_{z=0} \tag{2.5}$$

The following expressions give the solution of the transport equation and the values of the first unknown coefficients of seris (2.5)

$$\rho_1(t) = \frac{1}{F_1}\left(\rho_{10} + \int_0^t F_1 F_2\, dt\right), \quad u_2(t) = \frac{\rho_1}{F_0} - \frac{F_2}{\rho_0} \tag{2.6}$$

$$F_0 = \frac{4M^2\gamma T_0^{\omega-1}}{3\,\mathrm{Re}}, \quad F_1 = \exp\left[\int_0^t \left(\frac{\rho_0}{F_0} - \frac{2\rho_0'}{\rho_0}\right)dt\right]$$

$$F_2 = \frac{\omega\rho_0 T_1 u_1}{T_0} - \frac{3\,\mathrm{Re}\,\rho_0^2}{4T_0^{\omega}}\left(u_0' + \frac{T_1}{M^2\gamma}\right), \quad \rho_{10} = \left.\frac{\partial\rho_0(x)}{\partial x}\right|_{x=x_0}$$

$$T_2(t) = \frac{\mathrm{Pr}\,\mathrm{Re}}{\gamma T_0^{\lambda}}[\rho_0 T_0' + (\gamma-1)\rho_0 T_0 u_1 - (\gamma-1)F_0 T_0 u_1^2] - \frac{\lambda T_1^2}{T_0}$$

The following coefficients of series (2.5) can be written in terms of the preceding coefficients using the procedure described above, with help of quadratures and recurrence formulas. Since $\varkappa$ is independent of ρ, it follows that T_2 is independent of ρ_1. Therefore a weak discontinuity at the contact surface will appear in the "joined" solution in T not earlier than in the third derivative.

If in any problem the conditions are stationary or tend, as $t \to \infty$, to some limit values, then the behaviour of the solution will, in this case, be described as "stationarization" /2/ or "stabilization" /11/.

Let the contact surface be a heat-insulated impermeable piston at rest: $T_1 = u_0 = 0$. Then $F_2(t) = 0$ and the behaviour of $\rho_1(t)$ and $u_2(t)$ will be described by the function $F_1(t)$. If $\rho_0(t)$, $T_0(t)$ tend with time to some constant values ρ_{00}, T_{00}, then the solution (2.5) will describe the process of stabilization (stationarization) of the flow beside the heat-insulated impermeable wall. When the corresponding functions are analytic (see Sect.1) for all t, then series (2.5) will converge in some neighbourhood of the surface (2.2) for all t.

In order to analyse the behaviour of ρ_1, u_2, T_2 as $t \to +\infty$, we can use the following approximate estimates. Let us assume that for large t we can represent the functions ρ_0 and T_0 in the form of series in inverse powers of $(t + t_0)$

$$\rho_0(t) = \rho_{00} + \rho_{02}/(t + t_0) + \ldots, \quad T_0(t) = T_{00} + T_{02}/(t + t_0) + \ldots \tag{2.7}$$

Then F_1 can also be represented in the form of a series in inverse powers of $(t + t_0)$. If we retain the first two terms in this expansion, we obtain the law describing how $\rho_1(t)$ tends to zero as t increases

$$\rho_1(t) \approx \rho_{10} \exp(-Bt)(t + t_0)^{-BC} \tag{2.8}$$
$$B = \rho_{00}/F_0 > 0, \quad C = \rho_{02}/\rho_{00} + (1 - \omega)\, T_{02}/T_{00}$$

If we assume that the quantities ρ_{02}/ρ_{00} and T_{02}/T_{00} are of the same order, then we can find that for some relations governing $\operatorname{sign}\rho_{02}$, $\operatorname{sign} T_{02}$ and ω the quantity C will be positive, and for some other relations it will be negative. In particular, if the function $\rho_0(t)$ decreases (increases) as $t \to \infty$ to ρ_{00}, and $T_0(t)$ increases (decreases) to T_{00}, then, when $\omega > 0$, then $C > 0$ $(C < 0)$ always.

At present we know of very few /1, 3/ exact and approximate analytic solutions of (1.1). If in the course of constructing the series (2.5) we take, as ρ_0, T_0, functions simpler than (2.7), it will be possible to analyse the structure of the coefficients of series (2.5) and write out the required number of first terms of the series in explicit form.

Assertion 1. If $\rho_0(t), u_0(t), T_0(t), T_1(t) = \text{const}$, $T_1 = u_0 = 0$, then

$$\rho_1 = \rho_{10}\eta, \quad \rho_2 = \rho_{20}\eta + 3\rho_{10}^2\rho_{00}^{-1}\eta^2, \quad \rho_3 = \rho_{30}\eta + \rho_{31}\eta^2 + \rho_{32}\eta^3 + \rho_{33}t\eta, \quad u_1 = 0, \quad u_2 = B\rho_1/\rho_{00}, \quad u_3 = B\rho_2/\rho_{00} \tag{2.9}$$
$$T_2 = 0, \quad T_3 = (\gamma - 1)\gamma^{-1} \Pr \operatorname{Re} T_{00}^{1-\lambda} B\rho_{10}\eta$$
$$f_k = \eta P_k(t, \eta), \quad k \geqslant 4; \quad \eta = \exp(-Bt)$$

i.e. $f_k(t)$ have a multiplier η and are polynomials in t, η with constant coefficients. Here ρ_{3i} $(1 \leqslant i \leqslant 3)$ are uniquely defined constants, $f_k(t)$ are components of the vector $\mathbf{U}_k$, the power of the polynomials P_k depends linearly on k, the coefficients of the polynomials (except ρ_{k0}) are uniquely defined from the recurrence formulas, different for the different components of the vector $\mathbf{U}_k$, and the constants ρ_{k0} are found using relation (2.4).

The assertion is proved by induction over k, using the equation for f_k in explicit form, and the proof is omitted because of its length. As in /9/, we can show that in this case the domain of convergence of the series (2.5) is given by the relation

$$\xi_1 |z| < M_2, \quad \xi_1 = \max\{1, \eta, |t|\}, \quad M_2 = \text{const} > 0 \tag{2.10}$$

The presence in f_k of the factor η improves, when $t > 0$, the practical convergence of the series.

Relations (2.8) and (2.9) can be used for an approximate description of the process of smoothing a small perturbation near the corresponding contact surface: since $\rho_1(t)$ represents the value of $\partial\rho/\partial z$ at the contact surface, therefore the increment in density $\Delta\rho = \rho(r, t) - \rho(0, t)$ at a distance r from the contact surface can be approximately given by $\Delta\rho \approx \rho_1(t)\, r$. If in the course of introducing the dimensionless variables ρ_*, u_*, T_* are represented by ρ_{00}, $[c_v(\gamma - 1) T_{00}\gamma]^{1/2}$, T_{00}, respectively, then $\mathrm{M} = 1$, $B = 3\operatorname{Re}/(4\gamma)$, i.e. the exponent B is inversely proportional to the viscosity. Therefore, the lower the viscosity (strictly positive), the faster is the process of smoothing out the small perturbations near the corresponding contact surface. We note that the above argument is based on an analysis not of all the coefficients of the series, but only of the first ones. Therefore, the argument is valid only as long as the perturbation in question lies in the part of the domain of convergence of the series, in which the first terms are dominant. From (2.9) it follows that this part of the domain of convergence is also given by relation (2.10), but at a different (smaller) value of the constant M_2.

When we use the representation (2.5) to solve a specific problems, we must remember that there is an arbitrariness in the functions a, ρ_0, T_0, T_1. Using these functions we obtain, from the condition for the velocity at the contact surface and from the necessary condition of solvability of the characteristic problem, unique expressions for u_0, u_1. Moreover, when $t = 0$, the distribution of the density or velocity can be used as the arbitrary function. After this, the distributions of the other two gas-dynamic parameters when $t = 0$ and also of $\mathbf{U}$ when $t > 0$, are restored uniquely. When describing the flow in an approximate manner, we can use finite segments of the series, and utilize several expansions in the neighbourhoods of the different contact characteristics. The properties of series (2.5) and the form of their actual coefficients can also be used when constructing difference schemes near a contact surface specified a priori, as well as one found in the course of constructing the solution.

3. Let us give the solutions of system (2.1) in another form, based on the functions

η given above, and discuss its role in the process of stationarization.

We make the following change of variables in system (2.1):

$$\eta = \exp(-Bt), \quad x' = x \tag{3.1}$$

Here $\partial/\partial t = -B\eta\partial/\partial\eta$, and the rest of system (2.1) remains unchanged (the prime on x is omitted). We shall call the resulting system system (2.1) in the variables η, x. The Jacobian of the transformation (3.1) is equal to $-B\eta$, i.e. for $t_0 \leqslant t < +\infty$ the change of variables is in 1:1 correspondence. When $\eta = 0$, the transformation becomes degenerate because the infinite semi-axis is transformed into a segment of finite length.

We can write the solution of system (2.1) in variables η, x in the form

$$\mathbf{U}(\eta, x) = \sum_{k=0}^{\infty} \mathbf{U}_k(x) \frac{\eta^k}{k!} \tag{3.2}$$

To find the coefficients $\mathbf{U}_k(x)$ $(k \geqslant 0)$ we differentiate the system k times in η, and put $\eta = 0$. Here we find that $\mathbf{U}_0$ is a solution of the stationary Navier-Stokes system and corresponds to a limit flow to which the solution will tend as t increases.

When $\partial/\partial t = 0$, system (2.1) admits of three first integrals and is reduced to two first-order ordinary differential equations /1/. In particular, when $u_0 = 0$, we have either $\rho_0(x) = \rho_{00}$, $T_0(x) = T_{00}$, or $\rho_0(x) = C_1/T_0(x)$, $T_0(x) = (C_2 x + C_3)^{1/(1+\lambda)}$ $(\rho_{00}, T_{00}, C_1, C_2, C_3 = \mathrm{const})$. When $u_0 \neq 0$, we can take as U_0 e.g. the Bekker solution describing the passage through a shock /1/.

We obtain the following linear system of ordinary differential equations for $\mathbf{U}_k$ $(k \geqslant 1)$:

$$\frac{\partial}{\partial x}(u_0\rho_k) - kB\rho_k + \frac{\partial}{\partial x}(\rho_0 u_k) = F_{1k} \tag{3.3}$$

$$\rho_0(u_0' - kB)u_k + \rho_0 u_0 u_k' + u_0 u_0' \rho_k + \frac{1}{\mathrm{M}^2\gamma}\frac{\partial}{\partial x}(\rho_0 T_k + T_0\rho_k) =$$
$$\frac{4}{3\,\mathrm{Re}}\frac{\partial}{\partial x}(\omega T_0^{\omega-1}u_0' T_k + T_0^{\omega}u_k') + F_{2k}$$

$$\rho_0[(\gamma-1)u_0' - kB]T_k + \rho_0 u_0 T_k' + \rho_0 T_0' u_k + (\gamma-1)\rho_0 T_0 u_k' +$$
$$[u_0 T_0' + (\gamma-1)T_0 u_0']\rho_k =$$
$$(\gamma-1)\frac{4\mathrm{M}^2\gamma}{3\,\mathrm{Re}}(\omega T_0^{\omega-1}u_0'^2 T_k + 2T_0^{\omega}u_0' u_k') + \frac{\gamma}{\mathrm{Pr}\,\mathrm{Re}}\frac{\partial}{\partial x}(\lambda T_0^{\lambda-1}T_0' T_k + T_0^{\lambda}T_k') + F_{3k}$$

Here $F_{\alpha k}$ $(\alpha = 1, 2, 3)$ depends in a known manner on $\mathbf{U}_l$ $(0 \leqslant l \leqslant k-1)$ and on its derivatives. System (3.3) contains second-order derivatives of u_k, T_k as the leading derivatives. Therefore, in order to obtain unique values, we must specify two conditions for every u_k, T_k, which can be either the initial condition $x = 0$, or the boundary conditions for $x = 0$, $x = L$. Thus the representation (3.2) has two arbitrary functions of t, for u and for T. The arbitrary functions can be specified, either all of them for $x = 0$, or some for $x = 0$ and some for $x = L$. The first equation of system (3.3) has an arbitrariness, when $u_0 \neq 0$, in the choice of the constant specifying the value of ρ_k, e.g. when $x = 0$. If on the other hand $u_0 = 0$, the first equation of the system uniquely defines

$$\rho_k(x) = \left[\frac{\partial}{\partial x}(\rho_0 u_k) - F_{1k}\right](kB)^{-1}$$

In this case series (3.2) will show no arbitrariness in the choice of the function ρ.

Assertion 2. Let the solution $\mathbf{U}_0(x)$ of the stationary Navier-Stokes system be an analytic function in some neighbourhood of the point $x = 0$; the functions

$$x = 0,\ \rho = \rho_{01}(\eta),\ u = u_{01}(\eta),\ u_x = u_{02}(\eta)$$
$$T = T_{01}(\eta),\ T_x = T_{02}(\eta) \tag{3.4}$$

analytic in some neighbourhood of the point $\eta = 0$, match at $\eta = 0$ the values of $\mathbf{U}_0(0)$, $\mathbf{U}_{0x}(0)$; $\mathbf{U}_k(x)$ are found from systems (3.3), and the corresponding initial conditions are found from the functions (3.4). Then series (3.2) will converge in some neighbourhood of the point $(x = 0,\ \eta = 0)$, i.e. when $|x| < x^{\circ}$, $t^{\circ} \leqslant t < +\infty$.

A proof is not given here since the assertion represents a special case of a theorem proved in /8/. When $u_0 \neq 0$, all the functions (3.4) are used, while when $u_0 = 0$, the function ρ_{01} is not needed. We further write $\omega = \lambda$ and use, as the components of $\mathbf{U}_0$,

$$\rho_0(x) = \rho_{00},\ u_0(x) = 0,\ T_0(x) = T_{00} \tag{3.5}$$

When $k = 1$, we have $u_1 = T_1'(x)/(\mathrm{M}^2\gamma B)$ and system (3.3) reduces to a single homogeneous second-order equation for T_1. The roots of the characteristic equation for this differential equation are purely imaginary, therefore T_1 is a linear combination of the harmonics in x of frequency

$$\nu_1 = BMA_1 (\gamma/T_{00})^{1/2}, \ A_1 = [1 + \gamma\,(^3/_4\,\mathrm{Pr}^{-1} - 1)]^{-1/2} \tag{3.6}$$

Thu usual values for air are $\gamma = 1.4, \mathrm{Pr} = 0.72$ /1, 2/, and in this case $A_1 = 0.972$. The value $\mathrm{Pr} = {}^3/_4$ used in /1/ is taken sufficiently often in order to simplify the calculations, and in this case $A_1 = 1$.

When $k \geqslant 2$, the roots of the corresponding characteristic equation for system (3.3) are represented by four, purely imaginary, pairwise conjugate numbers. Therefore, the general solutions of the homogeneous systems for $\mathbf{U}_k$ are the corresponding linear combinations of the harmonics with frequencies

$$\nu_{k1} = k^{1/2}\nu_1, \ \nu_{k2} = k\nu_1\,[(k-1)\,\gamma]^{-1/2}, \ k \geqslant 2 \tag{3.7}$$

and both unknown functions contain harmonics of frequency ν_{k1}, as well as ν_{k2}. From (3.6) and (3.7) it follows that ν_{k1} and ν_{k2} increase monotonically as k; $\nu_1 < \nu_{k1}$; $\nu_1 < \nu_{k2}$ increases when $\gamma \leqslant \gamma_* = k^2/(k-1)$; the minimum value is $\gamma_* = 4$ and γ_* increases as k increases; when $\gamma > 1 + 1/(k-1)$, we have $\nu_{k2} < \nu_{k1}$ and, in particular, $\nu_{22} < \nu_{21}$ when $\gamma > 2$. Thus when $\gamma \leqslant 4$, the minimum value of the frequency is ν_1 and

$$\nu_2 = 2\nu_1\gamma^{-1/2} \tag{3.8}$$

when $\gamma > 4$. The harmonics of frequency ν_1 have a factor η the harmonics of frequency ν_2 a factor η^2, i.e. apart from the dependence on γ the latter decays more rapidly. When k and n are arbitrary, the frequencies $\nu_1, \nu_{k1}, \nu_{n1}, \nu_{k2}, \nu_{n2}$ will be pairwise incommensurable and the representation (3.2) will not, in general, be a periodic function.

We can, however, construct the following two classes of particular solutions. The first class is constructed as follows. If we take, as the components of the vector $\mathbf{U}_1$, the corresponding harmonics of frequency ν_1 and use, as the solutions of the inhomogeneous systems (3.3) with $k \geqslant 2$ only the particular solutions corresponding to the form of the right-hand sides, then $\mathbf{U}_k$ will be polynomials in h, xh. Here h is $\cos \nu_1 x, \sin \nu_1 x$ and the degree of these polynomials is not higher than k. Here x will appear for the first time (in the first power) in $\mathbf{U}_4$ when $\gamma \neq 4.5$, and in $\mathbf{U}_3$ when $\gamma = 4.5$. The second class of particular solutions is the following. If we take $\mathbf{U}_1 = 0$, then system (3.3) will be homogeneous when $k = 2$ and we can take the harmonics of frequency ν_2 as its solution. Then, if we take as $\mathbf{U}_k$ $(k \geqslant 3)$ the particular solutions of the inhomogeneous systems (3.3), then $\mathbf{U}_k$ will be polynomials of the same expressions as in the first case, where ν_1 should be replaced by ν_2. When γ, is not 2; 3,6; 4; 8; 10, then x will not appear in any power in $\mathbf{U}_k$ when $k \leqslant 10$. The domain of convergence of these particular solutions is given by the relation

$$\eta \mid x \mid^D < M_3; \ D, M_3 = \mathrm{const} > 0 \tag{3.9}$$

The classes of particular solutions constructed above are only slightly arbitrary - just the two constants in T_1 or in $\mathbf{U}_2$. The first of these classes can be extended, from the point of view of the arbitrariness within the solution, while retaining the structure of $\mathbf{U}_k$. To do this, we must take as $\mathbf{U}_k$, when $k = n^2$, not only the particular solutions of the inhomogeneous systems, but we must also add the general solutions of the inhomogeneous systems corresponding to the values ν_1. The $\mathbf{U}_k$ constructed in this manner will be polynomials in $x^l h^{m+1}$ where $l \geqslant 0, m \geqslant 0$, and $\mathbf{U}$ will contain a denumerable number of arbitrary constants. Without going into details of the proof, we note that if the moduli of these arbitrary constants increase not faster than the degree of some positive number, then the domain of convergence of the solutions belonging to such a class of particular solutions will also be given by relation (3.9) with its values of the constants D, M_3.

Thus we find, that for all three classes of particular solutions constructed the first terms of series (3.2) will be periodic functions of x of frequency ν_i ($i = 1$ for the first and third class, and $i = 2$ for the second class), and the subsequent terms will be functions oscillating with the same frequency. The oscillation amplitudes for the frequencies ν_i will decay as η^i, the values of ν_i will depend only on the parameters of the homogeneous limit flow $\mathbf{U}_0$, and the first terms of the series will be dominant in the corresponding parts of the domain of convergence. If, when introducing the dimensionless coordinates, we take as ρ_*, u_*, T_* (as was done above) the corresponding parameters of the limit flow (3.5), we will obtain the following proportionality relations for the index B, for the selected frequencies and for $L = 2\pi/\nu$:

$$B \sim 1/\mu, \ \nu_B \sim 1/\mu, \ L_B \sim \mu \tag{3.10}$$

We can expect that the classes of particular solutions of system (1.1) constructed here describe the viscous gas flows generated in homogeneous streams by the perturbations in which the main variation of the parameters takes place at distances commensurable with L_B.

As example of such a perturbation is a shock wave (SW) propagating through a homogeneous medium and leaving behind it a homogeneous flow. The width of such a SW is proportional to the coefficient of viscosity /1/. In a coordinate system moving together with SW, the latter appears as a constantly present external perturbation, and within its zone similar oscillations must appear all the time. Outside the SW zone, where there are no corresponding external perturbations, the amplitude of these oscillations should decay as $\exp(-Bt)$. It would appear that the small, rapid oscillatory displacements of the SW zone observed during experiments, and the oscillations appearing near the shock transitions (entropic trace) in the course of numerical solution of the flows, are caused precisely by the SW exciting similar oscillations within the flow.

For the fluid with $\mu = \mu_0 T^{\omega}$, the rise in temperature cuased by the passage of the gas across the SW, leads to an increase in viscosity, and hence reduces the rate of decay of the oscillations. Therefore the influence of the oscillations can manifest itself at small distances in front of the SW, and at relatively large distances behind it. For example, in the case of supersonic flows of a viscous gas past bodies, such oscillations excited by the bow SW may penetrate downstream in fairly large distances. They will therefore interact with various regions of the flow, namely with the boundary layer, the zones where the flow turns, etc. Such an interaction may, in its turn, cause the appearance of various instabilities.

The discussion following formulas (3.10) is obviously not rigorous, but more hypothetical in character, and the hypotheses need further checking both theoretically as well as experimentally.

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